

Inequalities and solution of an operator equation[☆]

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Abstract

In this work, we prove several important inequalities, and investigate the solution of an operator equation by means of the fixed point index in the theory of topological degree. For operator equations with different boundary conditions, the same conclusion is obtained. Meanwhile, the famous Rothe's Theorem is generalized. Finally, an example is given to show the application of Theorem 3.

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1. Introduction

Let E be a real Banach space. And let D be a bounded open convex set in E , ∂D the boundary of D in E .

A continuous operator $A : \bar{D} \rightarrow E$ is said to be a semi-closed 1-set-contractive operator if $I - A$ is a closed operator and $\alpha(AD) \leq \alpha(D)$, where α denotes the non-compact measure.

Throughout this work, some concepts are taken from Refs. [1–6].

In Ref. [1] Li established topological degree theorems for a semi-closed 1-set-contractive operator. In Ref. [3] Guo investigated some fixed point problems for nonlinear operators. In Ref. [7–12] Zhu generalized several famous theorems and investigated the theory of the fixed point index in functional analysis. In this work, we prove several important inequalities, and investigate a solution for the operator equation by means of the fixed point index in the theory of topological degree. Meanwhile, the famous Rothe's Theorem is generalized. Finally, an example is given to show the application of Theorem 3.

2. Main results

Lemma 1. *When $n > 1$, and $n \in N$, then the follow inequalities hold:*

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- (i) $t \geq e \ln t$, where $t > 0$;
(ii) $(t+1)^{n+1} + (t-1)^{n+1} > 2(t^{n+1} + 1)$, where $t > 1$.

Proof. Firstly, we prove that

$$t \geq e \ln t, \quad \text{where } t > 0. \quad (1)$$

Let $f(t) = t - e \ln t$, where $t > 0$; then $f'(t) = 1 - \frac{e}{t}$. Let $f'(t) = 0$, that is, $t = e$.

When $0 < t < e$, we have $f'(t) < 0$; when $e < t < +\infty$, we have $f'(t) > 0$. Thus, $f(e)$ is a minimum value of $f(t)$ in $(0, +\infty)$, that is, when $0 < t < +\infty$, we have $t \geq e \ln t$.

Secondly, we prove that

$$(t+1)^{n+1} + (t-1)^{n+1} > 2t^{n+1} + 2, \quad (2)$$

where $t > 1$, $n > 1$ and $n \in N$.

Let $g(t) = (t+1)^{n+1} + (t-1)^{n+1} - 2t^{n+1} - 2$, where $n > 1$, $n \in N$ and $t > 1$.

Then

$$\begin{aligned} g'(t) &= (n+1)(t+1)^n + (n+1)(t-1)^n - 2(n+1)t^n \\ &= (n+1)[(t+1)^n + (t-1)^n - 2t^n] \\ &= (n+1) \left[t^n + nt^{n-1} + \frac{n(n-1)}{2!}t^{n-2} + \cdots + 1 + t^n - nt^{n-1} + \frac{n(n-1)}{2!}t^{n-2} + \cdots + (-1)^n - 2t^n \right] \\ &= (n+1)\{n(n-1)t^{n-2} + \cdots [1 + (-1)^n]\} > 0. \end{aligned}$$

Hence, $g'(t) > 0$.

Therefore, $g(t)$ is a strictly monotone increasing function in $[1, +\infty)$.

Consequently, when $t > 1$, then $g(t) > g(1)$.

That is, $(t+1)^{n+1} + (t-1)^{n+1} > 2t^{n+1} + 2$, where $t > 1$, $n > 1$ and $n \in N$.

Theorem 1. Let D be a bounded open convex subset in E and $\theta \in D$. Suppose that $A : \bar{D} \rightarrow E$ is a semi-closed 1-set-contractive operator, and $Ax \neq x$, for every $x \in \partial D$. Meanwhile, the following condition holds:

$$(H_1) \quad \|Ax + x\|^{n+1} + \|Ax - x\|^{n+1} + \|x\| \leq 2[\|Ax\|^{n+1} + \|x\|^{n+1}] + e \ln \|x\|$$

for every $x \in \partial D$, $n > 1$ and $n \in N$; then the operator equation $Ax = x$ has a solution in D .

Proof. By the given conditions, we know that $Ax = x$ has no a solution in ∂D .

Let $H_t(x) = tAx$, where $t \in [0, 1]$, for every $x \in \bar{D}$; then $H_t : \bar{D} \rightarrow E$ is a semi-closed 1-set-contractive operator. In fact, we have $\alpha[H_t(D)] = \alpha[tA(D)] = t\alpha[AD] \leq t\alpha(D) \leq \alpha(D)$, where α denotes the non-compact measure.

Thus, $H_t : \bar{D} \rightarrow E$ is a 1-set-contractive operator. This is because

$$I - H_t = I - tA = I - tA + tI - tI = (1-t)I + t(I-A).$$

By virtue of the given conditions, we know that $H_t : \bar{D} \rightarrow E$ is a semi-closed operator.

Thus, we obtain that $H_t : \bar{D} \rightarrow E$ is a semi-closed 1-set-contractive operator. When $\|x\| > 0$, by (i) in Lemma 1, we have $e \ln \|x\| - \|x\| \leq 0$, and by (H₁) we have

$$\begin{aligned} \|Ax + x\|^{n+1} + \|Ax - x\|^{n+1} &\leq 2[\|Ax\|^{n+1} + \|x\|^{n+1}] + (e \ln \|x\| - \|x\|) \\ &\leq 2[\|Ax\|^{n+1} + \|x\|^{n+1}]. \end{aligned}$$

Hence, by the given condition (H₁) and Lemma 1, we can prove $x \neq H_t(x)$, for every $x \in \partial D$, $t \in [0, 1]$. According to the homotopy invariance property in Ref. [1], we know that

$$i(A, D, E) = i(\theta, D, E) = 1.$$

And by the solution property of Ref. [1], we know that $Ax = x$ has a solution in D .

Lemma 2. When $t > 1$, $p > 1$, the following inequality holds:

$$2(t-1)^p + 1 < t^p + (1+t)^p.$$

Proof. Let $f(t) = t^p - 1 - 2(t-1)^p + (1+t)^p$. Then

$$\begin{aligned} f'(t) &= pt^{p-1} - 2p(t-1)^{p-1} + p(1+t)^{p-1}, \\ &= p[t^{p-1} - 2(t-1)^{p-1} + (1+t)^{p-1}]. \end{aligned}$$

This is because

$$\begin{aligned} \frac{t^{p-1} + (1+t)^{p-1}}{-2(t-1)^{p-1}} &= \frac{t^{p-1}}{-2(t-1)^{p-1}} + \frac{(1+t)^{p-1}}{-2(t-1)^{p-1}} \\ &= -\frac{1}{2} \left(\frac{t}{t-1} \right)^{p-1} - \frac{1}{2} \left(\frac{t+1}{t-1} \right)^{p-1} \\ &= -\frac{1}{2} \left(\frac{t-1+1}{t-1} \right)^{p-1} - \frac{1}{2} \left(1 + \frac{2}{t-1} \right)^{p-1} \\ &< -\frac{1}{2} \left(1 + \frac{1}{t-1} \right)^{p-1} - \frac{1}{2} \left(1 + \frac{1}{t-1} \right)^{p-1} \\ &= -\left(1 + \frac{1}{t-1} \right)^{p-1} < -1. \end{aligned}$$

According to $-2(t-1)^{p-1} < 0$, we obtain that

$$\begin{aligned} t^{p-1} + (1+t)^{p-1} &> 2(t-1)^{p-1}, \quad \text{that is,} \\ t^{p-1} - 2(t-1)^{p-1} + (1+t)^{p-1} &> 0, \quad \text{that is,} \\ p[t^{p-1} - 2(t-1)^{p-1} + (1+t)^{p-1}] &> 0. \end{aligned}$$

Thus, $f'(t) > 0$.

Therefore $f(t)$ is a strictly monotone increasing function in $[1, +\infty)$. When $t > 1$, we have $f(t) > f(1)$ and $f(1) = 2^p > 0$.

Hence $f(t) > f(1) > 0$, that is, $f(t) > 0$, that is, $t^p + (1+t)^p > 2(t-1)^p + 1$, where $t > 1$, $p > 1$.

Theorem 2. Let D be a bounded open convex subset in E , and $\theta \in D$; suppose that $A : \bar{D} \rightarrow E$ is a semi-closed 1-set-contractive operator such that

$$(H_2) \quad 2\|Ax - x\|^p - \|x + Ax\|^p \geq \|Ax\|^p - \|x\|^p, \text{ for every } x \in \partial D, p > 1.$$

Then the operator equation $Ax = x$ has a solution in D .

Proof. By (H_2) , we know that $Ax = x$ has no solution in ∂D , that is,

$$x \neq Ax, \quad \text{for every } x \in \partial D. \quad (3)$$

We prove

$$x \neq tAx, \quad \text{for every } t \in (0, 1), \text{ for every } x \in \partial D. \quad (4)$$

In fact, suppose that (4) is not true, that is there exists a $t_0 \in (0, 1)$, and an $x_0 \in \partial D$ such that

$$x_0 = t_0 Ax_0, \quad \text{that is,} \quad Ax_0 = \frac{1}{t_0} x_0.$$

Inserting $Ax_0 = \frac{1}{t_0} x_0$ into (H_2) , we obtain

$$2 \left\| \frac{1}{t_0} x_0 - x_0 \right\|^p - \left\| x_0 + \frac{1}{t_0} x_0 \right\|^p \geq \left\| \frac{1}{t_0} x_0 \right\|^p - \|x_0\|^p, \quad \text{where } p > 1, x_0 \in \partial D,$$

that is,

$$2 \left\| \left(\frac{1}{t_0} - 1 \right) x_0 \right\|^p - \left\| \left(1 + \frac{1}{t_0} \right) x_0 \right\|^p \geq \left[\left(\frac{1}{t_0} \right)^p - 1 \right] \|x_0\|^p. \quad (5)$$

This is because $x_0 \in \partial D$; hence $\|x_0\| \neq 0$.

Therefore (5) gives that

$$2\left(\frac{1}{t_0} - 1\right)^p - \left(1 + \frac{1}{t_0}\right)^p \geq \left(\frac{1}{t_0}\right)^p - 1. \quad (6)$$

Let $t = \frac{1}{t_0}$; as $t_0 \in (0, 1)$, we obtain that $t > 1$. Hence (6) gives that

$$2(t-1)^p - (1+t)^p \geq (t)^p - 1, \quad (7)$$

that is, $2(t-1)^p + 1 \geq t^p + (1+t)^p$. This is a contradiction to Lemma 2.

Thus,

$$x \neq tAx, \quad \text{for every } x \in \partial D, t \in (0, 1). \quad (8)$$

By (3) and (8), we know that $x \neq tAx, t \in (0, 1]$, for every $x \in \partial D$.

By Ref. [1], we obtain that $i(A, D, E) = 1$. That is, the operator equation $Ax = x$ has a solution in D .

We can prove that the following Theorem 3 holds in an analogous way.

Theorem 3. Let D be a bounded open convex subset in E , and $\theta \in D$. Suppose that $A : \bar{D} \rightarrow E$ is a semi-closed l -set-contractive operator, and satisfies the following condition:

$$(H_3) \quad \|Ax - x_0\| \leq \|x - x_0\|, \quad \text{for every } x \in \partial D, \text{ and } x_0 \in D.$$

Then the operator equation $Ax = x$ has a solution in \bar{D} .

Remark 1. Theorem 3 generalizes the famous Rothe's Theorem [4,7].

Example 1. Let us consider the following integral equation which comes from information science and applied mathematics:

$$\int_0^x \left(\frac{1}{2} \sin |t| + \frac{1}{4} \cos |t| \right) dt - x + 2.1 = 0, \quad \forall x \in [-\pi, \pi]. \quad (9)$$

It is easy to prove that this equation has a solution in $[-\pi, \pi]$.

In fact, let

$$Ax = \int_0^x \left(\frac{1}{2} \sin |t| + \frac{1}{4} \cos |t| \right) dt + 2.1.$$

We write $\|y\| = |y|$, for every $y \in R$. Thus, we have

$$\begin{aligned} |A(-\pi) - 2.1| &= \left| \int_0^{-\pi} \left(\frac{1}{2} \sin |t| + \frac{1}{4} \cos |t| \right) dt \right| \\ &= \left| - \int_{-\pi}^0 \left(\frac{1}{2} \sin |t| + \frac{1}{4} \cos |t| \right) dt \right| \\ &= \left| \int_{-\pi}^0 \left(\frac{1}{2} \sin |t| + \frac{1}{4} \cos |t| \right) dt \right| \\ &\leq \int_{-\pi}^0 \left| \frac{1}{2} \sin |t| + \frac{1}{4} \cos |t| \right| dt \\ &\leq \int_{-\pi}^0 \left(\frac{1}{2} + \frac{1}{4} \right) dt \\ &= \frac{3}{4}\pi \\ &< |-\pi - 2.1| = \pi + 2.1. \end{aligned}$$

And

$$\begin{aligned}
 |A(\pi) - 2.1| &= \left| \int_0^\pi \left(\frac{1}{2} \sin |t| + \frac{1}{4} \cos |t| \right) dt \right| \\
 &= \left| \int_0^\pi \left(\frac{1}{2} \sin t + \frac{1}{4} \cos t \right) dt \right| \\
 &= \left| \frac{1}{2} \int_0^\pi \sin t dt + \frac{1}{4} \int_0^\pi \cos t dt \right| \\
 &= \left| \frac{1}{2} (-\cos t) \right|_0^\pi + \left| \frac{1}{4} (\sin t) \right|_0^\pi \\
 &= 1 < |\pi - 2.1| = \pi - 2.1.
 \end{aligned}$$

Let $D = [-\pi, \pi]$. It follows that $|Ax - 2.1| \leq |x - 2.1|$, for every $x \in \partial D$.

That is, $\|Ax - 2.1\| \leq \|x - 2.1\|$, for every $x \in \partial D$. Meanwhile, A is a semi-closed 1-set-contractive operator. According to [Theorem 3](#), we obtain that $Ax = x$ has a solution in $[-\pi, \pi]$. That is, [Eq. \(9\)](#) has a solution in $[-\pi, \pi]$.

References

- [1] Guozhen Li, The fixed point index and the fixed point theorems of 1-set-contractive mappings, *Proc. Amer. Math. Soc.* 104 (4) (1988) 1163–1170.
- [2] H. Aumann, On the number of solutions of nonlinear equation in ordered Banach spaces, *J. Funct. Anal.* 11 (2) (1972) 346–384.
- [3] Dajun Guo, *Nonlinear Functional Analysis*, Shandong Sci Tech Press, Ji'nan, 1985 (in Chinese).
- [4] Shisheng Zhang, *Fixed Point Theory and Application*, Chongqing Press, Chongqing, 1984 (in Chinese).
- [5] J. Henderson, H. Wang, Positive solutions for a nonlinear eigenvalue problem, *J. Math. Anal. Appl.* 288 (1997) 252–259.
- [6] Chuanxi Zhu, On random solution of random operator equation, *Adv. Math.* 26 (5) (1997) 429–434 (in Chinese).
- [7] Chuanxi Zhu, Several theorems of random semi-closed 1-set-contractive operator, *Acta Math. Sinica* 42 (3) (1999) 501–504 (in Chinese).
- [8] Chuanxi Zhu, Some new fixed point theorems in probabilistic metric spaces, *Appl. Math. Mech.* 16 (2) (1995) 179–185.
- [9] Chuanxi Zhu, Some theorems in the X–M–PN space, *Appl. Math. Mech.* 21 (2) (2000) 181–184.
- [10] Chuanxi Zhu, Some theorems for random operator equation of 1-set-contractive type, *Adv. Math.* 27 (5) (1998) 464–468 (in Chinese).
- [11] Chuanxi Zhu, Some problems in the Z–C–X space, *Appl. Math. Mech.* 23 (8) (2002) 511–514.
- [12] Chuanxi Zhu, Zongben Xu, Some theorems of random operator equations, *Internat. J. Math. Math. Sci.* 30 (9) (2002) 942–947.